Class 2, given on Jan 6, 2010, for Math 13, Winter 2010

In today's class we will quickly review the content of Chapter 13 in the text, which covers vectors and the geometry of two and three dimensional space. This is meant to be a refresher, not a detailed exposition of these topics, so do not expect extensive explanation or examples.

Much of this material will not appear in the first half of the class, but it will appear repeatedly in the second half.

1. Vectors

A *vector* can be thought of as an element of $\mathbb{R}^n$. However, we usually represent vectors as a pointed arrow, which has a tail at the origin and its arrowhead at the point described by the corresponding element of $\mathbb{R}^n$. Sometimes we permit vectors to start at points besides the origin. We commonly use either a boldface font, such as $\mathbf{v}$, or the notation $\vec{v}$ to distinguish vectors from real numbers.

We often write $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ to specify the *components* or *coordinates* of $\mathbf{v}$. If $n = 2, 3$, we may also use the notation $v_1\mathbf{i} + v_2\mathbf{j}$ or $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. In this notation, $\mathbf{i} = \langle 1, 0 \rangle$ or $\langle 1, 0, 0 \rangle$, etc.

We add and subtract two vectors by adding and subtracting the corresponding components. We can multiply a vector by a real number by multiplying each component by the real number in question. This is sometimes called *scalar multiplication*. Geometrically, the sum of two vectors can be thought of as one of the diagonals of the parallelogram determined by the two vectors.

A very important quantity associated to a vector is its *norm* or *length*. This is equal to the length of the arrow which represents the vector; if $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, then the norm, which is represented as $|\mathbf{v}|$ or $||\mathbf{v}||$ is given by the formula

$$|\mathbf{v}| = \sqrt{v_1^2 + \ldots + v_n^2}.$$

This can be proven by a straightforward application of the Pythagorean theorem.

If a vector has length 1, we call that vector a *unit vector*. Such vectors are very important; for example, when we calculate directional derivatives we use unit vectors. We will see more places where unit vectors appear in the second half of the class.

Example. A typical calculation you should expect to encounter relatively frequently is to find a unit vector which points in the same direction as some given vector. For example, suppose you are given the vector $\mathbf{v} = \langle 1/2, 1/3, 1/4 \rangle$. In general, a unit vector which points in the same direction as $\mathbf{v}$ is given by the formula

$$\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Notice that calculating the length of $\langle 1/2, 1/3, 1/4 \rangle$ is relatively unpleasant. A nice trick to simplify the calculation of unit vectors is to sometimes multiply or divide by a positive real number to simplify the form of the coordinates of the vector in question. For example, here, we can clear all the denominators by multiplying $\mathbf{v}$ by 12 to get $12\mathbf{v} = \langle 6, 4, 3 \rangle$. Then the unit vector which points in the same direction as $12\mathbf{v}$, which is the same as the unit vector pointing in the same direction as $\mathbf{v}$, is

$$\frac{1}{\sqrt{61}}\langle 6, 4, 3 \rangle.$$

This calculation is slightly simpler than calculating the length of $\mathbf{v}$ from directly, as you can see for yourself.

2. The dot product

Given two vectors $\mathbf{v} = \langle v_1, \ldots, v_n \rangle, \mathbf{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, their dot product, written $\mathbf{v} \cdot \mathbf{w}$, is defined to be the real number

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \ldots + v_n w_n.$$

One of the main applications of the dot product is that it gives us information about the angle between two vectors. As a matter of fact, an application of the law of cosines (which you may or may not remember from high school trigonometry) shows that if the angle between $\mathbf{v}, \mathbf{w}$ is θ, then

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}.$$

In particular, if the dot product of two vectors is zero, then they are perpendicular (orthogonal, normal) to each other.

Also notice that the length of a vector has a convenient expression in terms of the dot product:

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

The dot product can also be used to calculate the projection of one vector onto another, but we skip that application. See the text for more details.

3. The cross product

Now suppose $\mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are two vectors in $\mathbb{R}^3$. Their cross product, $\mathbf{v} \times \mathbf{w}$, is defined to be the vector

$$\langle v_2 w_3 - v_3 w_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - w_1 v_2 \rangle$$

You can either memorize this direction directly (you do need to know how to calculate cross products in this class), or use the fact that a cross product can be written as a 'determinant':

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

We sometimes use $|$ to delimit an array of numbers if we want to indicate the determinant of the corresponding matrix. The determinant of a 3×3 matrix is defined inductively in terms of the determinants of 2×2 matrices:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k}.$$

Make sure that the sign of the middle term is negative, if you choose to remember cross products this way. Recall that the determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Example. Calculate the cross product $\vec{v} \times \vec{w}$ of $\vec{v} = \langle 1, 2, -1 \rangle$ and $\vec{w} = \langle 3, -2, 1 \rangle$. We calculate the determinant of the matrix

$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix} = (2\cdot 1-(-2)\cdot(-1))\mathbf{i}-(1\cdot 1-(-1)\cdot 3)\mathbf{j}+(1\cdot(-2)-2\cdot 3)\mathbf{k} = -4\mathbf{j}-8\mathbf{k}.$$

The cross product of two vectors has some useful properties. For example, it is a vector which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. We list a few more properties below:

- The cross product is anti-commutative: $\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v}$.
- The cross product of two parallel vectors is $\mathbf{0}$.
- The magnitude of $\mathbf{v}\times\mathbf{w}$ is equal to $|\mathbf{v}||\mathbf{w}|\sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. By trigonometry, this is equal to the area of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$.
- There are two possible directions for $\mathbf{v}\times\mathbf{w}$, since it is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. The direction is given by what is known as the *right hand rule*. If we take the all the fingers in our right hand, except our thumb, and curl them in the direction of $\mathbf{v}$ to $\mathbf{w}$, then the thumb points in the direction of $\mathbf{v}\times\mathbf{w}$.

Example. The arrangement of the x, y, and z axes in $\mathbb{R}^3$ is setup so that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$. Another example: suppose we draw $\mathbf{v}$, which is pointing left on a blackboard, and $\mathbf{w}$, which is pointing down and to the right. Then $\mathbf{v}\times\mathbf{w}$ will be pointing out of the blackboard (as opposed to in).

4. Lines and planes in $\mathbb{R}^3$

We know how to describe lines in $\mathbb{R}^2$. The point-slope and slope-intercept forms for such lines are common, with a parametric description of lines in $\mathbb{R}^2$ being slightly less common. In $\mathbb{R}^3$, however, the parametric description of a line is most convenient to use.

Consider a line in $\mathbb{R}^3$ (or any $\mathbb{R}^n$, for that matter). A line is completely determined by any two distinct points that lie on it. However, this is the same as saying that a line is completely determined by a point on it, and a *direction vector* for that line, where a direction vector is any vector whose endpoints are two distinct points on the line.

If (x_0, y_0, z_0) is any point on a line ℓ, and $\langle a, b, c\rangle$ a direction vector for ℓ, then the line is given in parametric form by

$$\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned}$$

This parametric representation of a line is far from unique. We can choose any point which lies on ℓ for (x_0, y_0, z_0), and any nonzero scalar multiple of a direction vector for ℓ is still a direction vector.

Example. Let $4x + y = 6$ be a line in $\mathbb{R}^2$. Let's find parametric equations for this line. We start by finding any two points on this line; for example, $(0, 6)$ and $(1, 2)$ will do. Then the direction vector whose endpoints are these two points is given by $\langle 1, -4\rangle$. Therefore, parametric equations for this line are given by

$$\begin{aligned} x(t) &= t \\ y(t) &= 6 - 4t \end{aligned}$$

If you want to check your work, you can substitute these equations back into the equation $4x + y = 6$ to check that you do not get a contradiction. Of course, these parametric equations are not unique, since different choices for the starting two points on the line will usually give different parametric equations.

In contrast to lines, a plane in $\mathbb{R}^3$ (this time, our discussion does not generalize to $\mathbb{R}^n$ as easily) is determined by three non-colinear points. However, it is not entirely obvious how one should get a simple equation from such a description. A bit of thought shows that a plane is also determined by any point which lies on a plane, and a *normal vector* for the plane. A normal vector for a plane is a vector which is orthogonal to every vector that lies on the plane.

If $\langle a, b, c\rangle$ is a normal vector for a plane which passes through (x_0, y_0, z_0), then an equation for the plane is given by

$$ax + by + cz = ax_0 + by_0 + cz_0 = d.$$

Again, a normal vector is not unique; any nonzero scalar multiple of a normal vector is also a normal vector. This form of an equation for a plane is often called an *implicit form*.

How can we translate a description which involves three non-colinear points into an equation as above? Suppose P, Q, R are three non-colinear points on a plane. Form the vectors $\vec{PQ}, \vec{PR}$. Non-colinearity ensures that these two vectors are not parallel. Then their cross product is orthogonal to both $\vec{PQ}, \vec{PR}$, and a bit of thought will show that the cross product will then be orthogonal to any vector on the plane. This cross product, which is nonzero, is a normal vector, and then we use any of the three points P, Q, R to calculate d.